

THE AVERAGE SIZES OF TWO-TORSION SUBGROUPS IN QUOTIENTS OF CLASS GROUPS OF CUBIC FIELDS

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ABSTRACT. We prove a generalization of a result of Bhargava regarding the average size $\text{Cl}(K)[2]$ as K varies among cubic fields. For a fixed set of rational primes S , we obtain a formula for the average size of $\text{Cl}(K)/\langle S \rangle[2]$ as K varies among cubic fields with a fixed signature, where $\langle S \rangle$ is the subgroup of $\text{Cl}(K)$ generated by the classes of primes of K above prime in S . We additionally obtain average sizes for the relaxed Selmer group $\text{Sel}_2^S(K)$ and for $\mathcal{O}_{K,S}^\times/(\mathcal{O}_{K,S}^\times)^2$ as K varies in the same families.

1. INTRODUCTION

In addition to the Davenport-Heilbronn theorem, one of the few results proven concerning the distribution of class groups of number fields is a result of Bhargava in [1] and extended by Bhargava and Varma in [4] which states:

Theorem 1. *When ordered by absolute discriminant,*

- (i) *the average size of $\text{Cl}(K)[2]$ as K ranges over totally real \mathcal{S}_3 -cubic fields is equal to $5/4$,*
- (ii) *the average size of $\text{Cl}(K)[2]$ as K ranges over complex \mathcal{S}_3 -cubic fields is equal to $3/2$,*
and
- (iii) *the average size of $\text{Cl}(K)^+[2]$ as K ranges over totally real \mathcal{S}_3 -cubic fields is equal to 2 .*

Theorem 1 may be thought of as an analogue of the classical Davenport-Heilbronn theorem regarding the average size of the 3-torsion subgroups of class groups of quadratic fields. In [10], we generalized the Davenport-Heilbronn theorem to quotients of ideal class groups of quadratic fields by the subgroup generated by the classes of primes lying above a fixed set of rational primes S . The goal of this work is to do the same for Theorem 1.

Explicitly: Let S be a finite set of rational primes. For each cubic field K , define $\text{Cl}(K)_S := \text{Cl}(K)/\langle S_K \rangle$, where S_K is the set of primes of \mathcal{O}_K lying above the primes in S and $\langle S_K \rangle$ is the subgroup of $\text{Cl}(K)$ generated by the ideal classes of the primes in S_K . Define $\text{Cl}(K)_S^+$ similarly using the narrow class group $\text{Cl}(K)^+$ of K .

Theorem 2. *When ordered by absolute discriminant,*

- (i) *the average size of $\text{Cl}(K)_S[2]$ as K ranges over totally real \mathcal{S}_3 -cubic fields is equal to*
$$1 + \frac{1}{2^{|S|+2}} \prod_{p \in S} \left(1 + \frac{p^2 + 4}{4(p^2 + p + 1)} \right),$$
- (ii) *the average size of $\text{Cl}(K)_S[2]$ as K ranges over complex \mathcal{S}_3 -cubic fields is equal to*
$$1 + \frac{1}{2^{|S|+1}} \prod_{p \in S} \left(1 + \frac{p^2 + 4}{4(p^2 + p + 1)} \right),$$
 and
- (iii) *the average size of $\text{Cl}(K)_S^+[2]$ as K ranges over totally real \mathcal{S}_3 -cubic fields is equal to*
$$1 + \frac{1}{2^{|S|}} \prod_{p \in S} \left(1 + \frac{p^2 + 4}{4(p^2 + p + 1)} \right).$$

Theorem 2 has the following immediate corollary.

Corollary 3. *If S is non-empty, then*

- (i) *a positive proportion of totally real \mathcal{S}_3 -cubic fields K have $\text{Cl}(K)_S^+[2] = 0$,*
- (ii) *a positive proportion of totally real \mathcal{S}_3 -cubic fields K have $\text{Cl}(K)_S^+ = \text{Cl}(K)$, and*
- (iii) *a positive proportion of totally real \mathcal{S}_3 -cubic fields K have S_K -units of all possible signatures.*

In each case, the proportion of totally real \mathcal{S}_3 -cubic fields having the property claimed is at least $1 - \frac{1}{2^{|S|}} \prod_{p \in S} \left(1 + \frac{p^2 + 4}{4(p^2 + p + 1)}\right)$.

The core idea behind Theorem 1 is how to use the geometry of numbers to count \mathcal{S}_4 -quartic fields. The application to class groups as it originally appears in [1] arises from a bijection established by Heilbronn and slightly refined by Bhargava between the set of non-trivial two-torsion elements in the class group $\text{Cl}(K)$ of an \mathcal{S}_3 -cubic field and the set of *nowhere overramified* isomorphism classes of \mathcal{S}_4 -quartic fields L with *resolvent field* K (see Definitions 2.1 and 2.2) that have a real place.

We establish a similar bijection between the set of non-trivial two-torsion elements in $\text{Cl}(K)_S$ and the set of isomorphism classes of \mathcal{S}_4 -quartic fields L with *resolvent field* K satisfying certain local conditions. We then prove Theorem 2 by applying recent results of Bhargava for counting the number of such fields having bounded discriminant [2].

The local product appearing in Theorem 2 is a consequence of the fact that the decomposition type of any prime in S will vary with K . By assuming a fixed decomposition type for each prime in S , we get a more natural answer.

Theorem 4. *Let S be a set of rational primes. For each $p \in S$, fix a rank 3 \mathbb{Q}_p -algebra R_p and set $r = \sum_{p \in S} (r_p - 1)$, where r_p is the number of irreducible components of R_p . When ordered by absolute discriminant,*

- (i) *the average size of $\text{Cl}(K)_S[2]$ as K ranges over totally real \mathcal{S}_3 -cubic fields with $K \otimes \mathbb{Q}_p \simeq R_p$ for all $p \in S$ is equal to $1 + 2^{-(r+2)}$,*
- (ii) *the average size of $\text{Cl}(K)_S[2]$ as K ranges over complex \mathcal{S}_3 -cubic fields with $K \otimes \mathbb{Q}_p \simeq R_p$ for all $p \in S$ is equal to $1 + 2^{-(r+1)}$, and*
- (iii) *the average size of $\text{Cl}(K)_S^+[2]$ as K ranges over totally real \mathcal{S}_3 -cubic fields with $K \otimes \mathbb{Q}_p \simeq R_p$ for all $p \in S$ is equal to $1 + 2^{-r}$*

Remark 1.1. In Theorem 4, r_p is the number of primes above p in K . If the primes above p take classes uniformly at random in $\text{Cl}(K)$ subject only to the relation arising from the factorization of $p\mathcal{O}_K$ and all of the primes in S behave independently, then the subgroup $\langle S_K \rangle \leq \text{Cl}(K)$ may be thought of as a group generated by r elements chosen uniformly at random from $\text{Cl}(K)$. It is therefore natural to expect that for any finite abelian 2-group H , the probability $\text{Prob}(\text{Cl}(K)/\langle S \rangle[2^\infty] \simeq H)$ is equal to what Cohen and Lenstra dub the u -probability of H with $u = r + r_\infty$, where r_∞ is equal to 2, 1, and 0 in cases (i), (ii), and (iii) of the theorem respectively. The average sizes appearing Theorem 4 are precisely the u -averages for these values of u [6].

In addition to Theorem 2, we prove similar distribution results for a pair of objects closely connected to $\text{Cl}(K)_S[2]$. A standard result (see Section 8.3.2 of [5], for example) shows that

$\text{Cl}(K)_S[2]$ sits in the short exact sequence

$$(1) \quad 0 \rightarrow \mathcal{O}_{K,S}^\times / (\mathcal{O}_{K,S}^\times)^2 \rightarrow \text{Sel}_2^S(K) \rightarrow \text{Cl}(K)_S[2] \rightarrow 0,$$

where $\mathcal{O}_{K,S}^\times$ is the S_K units of \mathcal{O}_K and $\text{Sel}_2^S(K)$ is the 2-Selmer group of K relaxed at S_K , defined as

$$\text{Sel}_2^S(K) := \{ \alpha \in K^\times / (K^\times)^2 : \text{val}_{\mathfrak{p}}(\alpha) \equiv 0 \pmod{2} \text{ for all } \mathfrak{p} \notin S_K \}.$$

We are able to compute average sizes for both $\mathcal{O}_{K,S}^\times / (\mathcal{O}_{K,S}^\times)^2$ and $\text{Sel}_2^S(K)$.

Theorem 5. *When ordered by absolute discriminant,*

(i) *the average size of $\text{Sel}_2^S(K)$ as K ranges over totally real \mathcal{S}_3 -cubic fields is equal to*

$$2^{|S|} + 2^{|S|+3} \prod_{p \in S} \left(2 - \frac{1}{p^2 + p + 1} \right) \text{ and}$$

(ii) *the average size of $\text{Sel}_2^S(K)$ as K ranges over complex \mathcal{S}_3 -cubic fields is equal to*

$$2^{|S|} + 2^{|S|+2} \prod_{p \in S} \left(2 - \frac{1}{p^2 + p + 1} \right).$$

Theorem 6. *When ordered by absolute discriminant,*

(i) *the average size of $\mathcal{O}_{K,S}^\times / (\mathcal{O}_{K,S}^\times)^2$ as K ranges over totally real \mathcal{S}_3 -cubic fields is equal*

$$\text{to } 2^{|S|+3} \prod_{p \in S} \left(2 - \frac{1}{p^2 + p + 1} \right) \text{ and}$$

(ii) *the average size of $\mathcal{O}_{K,S}^\times / (\mathcal{O}_{K,S}^\times)^2$ as K ranges over complex \mathcal{S}_3 -cubic fields is equal to*

$$2^{|S|+2} \prod_{p \in S} \left(2 - \frac{1}{p^2 + p + 1} \right).$$

Remark 1.2. All of the results we present are stated in terms of \mathcal{S}_3 -cubic fields. However, the results remain correct and can be proven with modified versions of the current proofs even if we remove the \mathcal{S}_3 assumption.

We have nonetheless chosen to maintain the \mathcal{S}_3 -assumption throughout the paper for the purposes of clarity. Instead, we present the following argument showing that the total contribution to the average size of $\text{Cl}(K)[2]$ coming from cyclic cubic fields as K varies among all totally real cubic fields is zero.

It is well-known that the total number of cyclic cubic fields of discriminant less than X grows like $O(X^{1/2})$ [7]. By a result of Wong [11], the size of $\text{Cl}(K)[2]$ in any cyclic cubic field K is bounded by $O(|\text{Disc}(K)|^{3/8+\epsilon})$ for any $\epsilon > 0$ (see also [3] for a better bound). As a result, the combined number of elements in $\text{Cl}(K)[2]$ from all cyclic cubic fields with discriminant less than X is bounded by $O(X^{7/8+\epsilon})$. Since the number of totally real cubic fields of discriminant less than X grows like $O(X)$, the total contribution to the average size of $\text{Cl}(K)[2]$ coming from cyclic cubic fields must be zero.

NOTATION

We will use the following notation throughout this paper:

- S will be a set of rational primes.
- K will be a cubic field.
- \mathcal{O}_K will be the ring of integers of K .

- S_K will denote the set of primes of \mathcal{O}_K lying above primes in S .
- $\mathcal{O}_{K,S}^\times$ will denote the S_K -units of K .
- $\text{Sel}_2^S(K)$ will be the 2-Selmer group of K relaxed at the primes in S_K .
- $\text{Cl}(K)$ will be the ideal class group of \mathcal{O}_K .
- $\text{Cl}(K)_S$ will be the quotient $\text{Cl}(K)/\langle S_K \rangle$, where $\langle S_K \rangle$ is the subgroup of $\text{Cl}(K)$ generated by primes in S_K .

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2. CLASS FIELDS THEORY

Bhargava's results about $\text{Cl}(K)[2]$ in [1] rely on a correspondence of Heilbronn detailed in [9]. We briefly describe this correspondence before establishing a similar correspondence for $\text{Cl}(K)_S[2]$ in Proposition 2.6.

Definition 2.1. Given an \mathcal{S}_4 -quartic field L , the *cubic resolvent field* $\text{Res}(L)$ is the unique (up to isomorphism) cubic subfield of the Galois closure N of L/\mathbb{Q} .

While the resolvent field of L is unique, non-isomorphic L may share the same resolvent field.

Following Section 3.1 in [1], we make the following definition.

Definition 2.2. Let p be a rational prime. A quartic field L is called *overramified at p* if $L \otimes \mathbb{Q}_p$ is either irreducible and ramified or the direct sum of two ramified fields. The field L is called *nowhere overramified* if L is not overramified at p for any prime p .

Remark 2.3. Note that unlike Bhargava's definition in [1], our definition of nowhere overramified does not include any restriction on the ramification of L at infinity.

The fields L and $\text{Res}(L)$ have the same discriminant precisely when L is nowhere overramified.

Proposition 2.4. *Let K be an \mathcal{S}_3 -cubic field.*

- (A) *The following are in bijective correspondence.*
- (i) *The set of index two subgroups of $\text{Cl}(K)$.*
 - (ii) *The set of unramified quadratic extensions F of K .*
 - (iii) *The set of isomorphism classes of nowhere overramified \mathcal{S}_4 -quartic fields L having a real place with $\text{Res}(L) = K$.*
- (B) *The following are in bijective correspondence.*
- (i) *The set of index two subgroups of $\text{Cl}(K)^+$.*
 - (ii) *The set of quadratic extensions F of K that are unramified at all finite places.*
 - (iii) *The set of isomorphism classes of nowhere overramified \mathcal{S}_4 -quartic fields L with $\text{Res}(L) = K$.*

Proof. For both (A) and (B), the correspondence (i) \leftrightarrow (ii) is class field theory. The correspondence between (ii) and (iii) in (B) is detailed in [9] and the correspondence between (ii) and (iii) in (A) comes from restricting this correspondence to the sets in (A). While we do

not include a proof of the equivalence between (ii) and (iii) here, we will however describe the maps yielding the correspondence.

We begin with the map (iii) \rightarrow (ii). Let N be the Galois closure of L/\mathbb{Q} . By assumption, $\text{Gal}(N/\mathbb{Q}) \simeq \mathcal{S}_4$. The group \mathcal{S}_4 contains three distinct D_4 subgroups, all of which are conjugate. The resolvent field $K = \text{Res}(L)$ may be taken to be the fixed field of any of these D_4 subgroups. The group D_4 contains two distinct Klein four subgroups, only one of which contains a transposition when D_4 is viewed as a subgroup of $\text{Gal}(N/\mathbb{Q}) \simeq \mathcal{S}_4$. Letting V be that subgroup, the extension F/K is given by the fixed field of V .

We next describe the map (ii) \rightarrow (iii). While the field F/\mathbb{Q} is not Galois, the lack of ramification (at finite primes) in F/K forces its Galois closure N to be an \mathcal{S}_4 -extension of \mathbb{Q} . The group \mathcal{S}_4 contains four distinct \mathcal{S}_3 subgroups, all of which are conjugate. The field L may be taken to be the fixed field of any of these \mathcal{S}_3 subgroups. \square

Remark 2.5. In both Proposition 2.4 and Proposition 2.6 which follows, if K is taken to be a complex cubic field, then $\text{Cl}(K) = \text{Cl}(K)^+$ and parts (A) and (B) are equivalent.

Proposition 2.6. *Let K be an \mathcal{S}_3 -cubic field.*

(A) *The following are in bijective correspondence.*

- (i) *The set of index two subgroups of $\text{Cl}(K)_S$.*
- (ii) *The set of unramified quadratic extensions F of K in which all primes in S_K split completely.*
- (iii) *The set of isomorphism classes of nowhere overramified \mathcal{S}_4 -quartic fields L such that $\text{Res}(L) = K$, L has a real place, and $L \otimes \mathbb{Q}_p$ has a component equal to \mathbb{Q}_p for all $p \in S$.*

(B) *The following are in bijective correspondence.*

- (i) *The set of index two subgroups of $\text{Cl}(K)_S^+$.*
- (ii) *The set of quadratic extensions F of K that are unramified at all finite places and in which all primes in S_K split completely.*
- (iii) *The set of isomorphism classes of nowhere overramified \mathcal{S}_4 -quartic fields L such that $\text{Res}(L) = K$ and $L \otimes \mathbb{Q}_p$ has a component equal to \mathbb{Q}_p for all $p \in S$.*

Proof. For both (A) and (B), the correspondence (i) \leftrightarrow (ii) is class field theory. The equivalences (ii) \leftrightarrow (iii) in (A) and (B) follow from the similar equivalences in Proposition 2.4 and Lemma 2.7 appearing below. \square

Lemma 2.7. *Let K be an \mathcal{S}_3 -cubic field and F/K a quadratic extension unramified at all finite places. If L is the \mathcal{S}_4 -quartic field corresponding to F/K under Heilbronn's correspondence as in Proposition 2.4, then $L \otimes \mathbb{Q}_p$ has a component equal to \mathbb{Q}_p if and only if all primes of K above p split in F/K .*

Proof. Let N be the Galois closure of L/\mathbb{Q} . If $L \otimes \mathbb{Q}_p$ has a component equal to \mathbb{Q}_p , then $\text{Gal}(N_P/\mathbb{Q}_p) \leq \mathcal{S}_3$ for all primes $P \mid p$ of N . Letting \mathfrak{p} be a prime of K above p , we have $\text{Gal}(N_P/K_{\mathfrak{p}}) \leq \text{Gal}(N/K)$. Since $\text{Gal}(N_P/K_{\mathfrak{p}})$ must simultaneously embed into a copy of \mathcal{S}_3 and a copy of D_4 inside of \mathcal{S}_4 , we find that $\text{Gal}(N_P/K_{\mathfrak{p}})$ must be a subgroup of the Klein four group $\text{Gal}(N/F)$. As a result, \mathfrak{p} splits in F/K .

For the opposite direction, we rely on the Artin relation of zeta functions (see [9], for example)

$$(2) \quad \zeta_L(s) = \frac{\zeta(s)\zeta_F(s)}{\zeta_K(s)}.$$

If all primes of K above p split in F/K , then we have $\zeta_F^{(p)}(s) = \zeta_K^{(p)}(s)$, where $\zeta_*^{(p)}(s)$ denotes the part of the Euler product for $\zeta_*(s)$ coming from primes above p . As a result, if all primes of K above p split in F/K , then (2) yields

$$(3) \quad \zeta_L^{(p)}(s) = (1 - p^{-s})^{-1} \zeta_K^{(p)}(s).$$

Observe that if $L \otimes \mathbb{Q}_p$ contains a ramified component, then $K \otimes \mathbb{Q}_p$ must contain a ramified component of the same degree. As a result, (3) holds even if we restrict to the Euler factors coming from unramified primes. As a result, the Euler product for $\zeta_L(s)$ contains a factor of $(1 - p^{-s})^{-1}$ coming from an unramified prime above p , and the $L \otimes \mathbb{Q}_p$ has a component equal to \mathbb{Q}_p . \square

We therefore get the following corollary:

Corollary 2.8. *If K is an \mathcal{S}_3 -cubic field, then*

$$(i) \quad |\mathrm{Cl}(K)_S[2]| = 1 + \left| \left\{ \begin{array}{l} \text{nowhere overramified } \mathcal{S}_4\text{-quartic fields } L \text{ (up to iso.)} \\ \text{such that } \mathrm{Res}(L) = K, L \text{ has a real place, and} \\ L \otimes \mathbb{Q}_p \text{ has a component equal to } \mathbb{Q}_p \text{ for all } p \in S \end{array} \right\} \right| \text{ and}$$

$$(ii) \quad |\mathrm{Cl}(K)_S^+[2]| = 1 + \left| \left\{ \begin{array}{l} \text{nowhere overramified } \mathcal{S}_4\text{-quartic fields } L \\ \text{(up to iso.) such that } \mathrm{Res}(L) = K \text{ and } L \otimes \mathbb{Q}_p \\ \text{has a component equal to } \mathbb{Q}_p \text{ for all } p \in S \end{array} \right\} \right|.$$

Proof. Since $\mathrm{Cl}(K)_S$ is a finite abelian group, the number of index two subgroups of $\mathrm{Cl}(K)_S$ is the same as the number of non-trivial two-torsion elements of $\mathrm{Cl}(K)_S$. By Proposition 2.6, this is equal to the number of quartic fields L such that $\mathrm{Res}(L) = K$, L has a real place and $L \otimes \mathbb{Q}_p$ has \mathbb{Q}_p component for all $p \in S$. The result for $\mathrm{Cl}(K)_S^+$ follows similarly. \square

3. COUNTING FIELDS

In order to prove Theorems 2 and 4, we will need to be able to count the number of quartic fields of bounded discriminant satisfying a given set of local conditions.

For a set Σ_p of \mathbb{Q}_p -algebras, define $\mu_p(\Sigma_p)$ as

$$\mu_p(\Sigma_p) := \sum_{R \in \Sigma_p} \frac{p-1}{p} \cdot \frac{1}{\mathrm{Disc}_p(R)} \cdot \frac{1}{|\mathrm{Aut}(R)|},$$

where $\mathrm{Disc}_p(R)$ is the p -part of the discriminant of R .

For each $p \in S$, let Σ_p be a set of non-overramified rank 4 \mathbb{Q}_p and set $\Sigma = (\Sigma_p)_{p \in S}$. For each $i \in \{0, 2, 4\}$, define $N^{(i)}(X, \Sigma)$ to be the number of nowhere overramified \mathcal{S}_4 -quartic fields L (up to isomorphism) such that $|\mathrm{Disc}(L)| < X$, L has i real places, and $L \otimes \mathbb{Q}_p \in \Sigma_p$ for all $p \in S$.

We then have the following specialization of a theorem of Bhargava.

Theorem 3.1 (Theorem 1.3 in [2]). *For each $i \in \{0, 2, 4\}$,*

$$(4) \quad N^{(i)}(X, \Sigma) = \frac{1}{2n_i \zeta(3)} \prod_{p \in S} \frac{\mu_p(\Sigma_p)}{\mu_p} \cdot X + o(X)$$

where $\mu_p = 1 - \frac{1}{p^3}$ and $n_i = \begin{cases} 8 & \text{if } i = 0 \\ 4 & \text{if } i = 2 \\ 24 & \text{if } i = 4 \end{cases}$.

Proof. This will follow from Theorem 1.3 in [2]. For each prime p , define $\hat{\Sigma}_p$ to be the set of all rank 4 non-overramified \mathbb{Q}_p -algebras. An easy computation shows that $\mu_p(\Sigma_p) = 1 - \frac{1}{p^3}$. We then set $\hat{\Sigma} = (\hat{\Sigma}_p)_p$.

For each prime $p \in S$, let Σ_p be as above and for $p \notin S$, set $\Sigma_p = \hat{\Sigma}_p$. Applying Theorem 1.3 in [2], we then get

$$N^{(i)}(X, \Sigma) = \frac{1}{n_i} \prod_p \mu_p(\hat{\Sigma}_p) \prod_{p \in S} \frac{\mu_p(\Sigma_p)}{\mu_p(\hat{\Sigma}_p)} = N^{(i)}(X, \hat{\Sigma}) \prod_{p \in S} \frac{\mu_p(\Sigma_p)}{\mu_p(\hat{\Sigma}_p)}$$

However, $N^{(i)}(X, \hat{\Sigma})$ is simply the number of nowhere totally ramified \mathcal{S}_4 -quartic fields L (up to isomorphism) such that $|\text{Disc}(L)| < X$ and L has i real places. By Lemma 27 in [1], this is known to be $\frac{1}{2n_i\zeta(3)} \cdot X + o(X)$, so the result follows. \square

We may similarly count the number of cubic fields of bounded discriminant satisfying a set of local conditions. For a set of local conditions $\Sigma = (\Sigma_p)_{p \in S}$ where each Σ_p is a set of rank 3 \mathbb{Q}_p -algebras and each $i \in \{1, 3\}$, define $M^{(i)}(X, \Sigma)$ to be the number of \mathcal{S}_3 -cubic fields K (up to isomorphism) such that $|\text{Disc}(K)| < X$, K has i real places, and $K \otimes \mathbb{Q}_p \in \Sigma_p$ for all $p \in S$.

Theorem 3.2 (Theorem 1.3 in [2]). *For each $i \in \{1, 3\}$,*

$$(5) \quad M^{(i)}(X, \Sigma) = \frac{1}{2m_i\zeta(3)} \prod_{p \in S} \frac{\mu_p(\Sigma_p)}{\boldsymbol{\mu}_p} \cdot X + o(X)$$

where $m_1 = 2$, $m_3 = 6$, and $\boldsymbol{\mu}_p$ is as in Theorem 3.1.

The proof of Theorem 3.2 is extremely similar to that of Theorem 3.1 so we omit it.

4. DEALING WITH LOCAL CONDITIONS

In general, if K is an \mathcal{S}_3 -cubic field and L an \mathcal{S}_4 -quartic field such that the resolvent $\text{Res}(L) = K$, then $K \otimes \mathbb{Q}_p$ does not determine $L \otimes \mathbb{Q}_p$. However, if we further assume that $L \otimes \mathbb{Q}_p$ has a \mathbb{Q}_p component, then this is no longer the case.

Lemma 4.1. *Let K be an \mathcal{S}_3 -cubic field and L an \mathcal{S}_4 -quartic field such that the resolvent $\text{Res}(L) = K$. If p is a prime such that $L \otimes \mathbb{Q}_p$ has a component equal to \mathbb{Q}_p , then $L \otimes \mathbb{Q}_p \simeq (K \otimes \mathbb{Q}_p) \oplus \mathbb{Q}_p$.*

Proof. Let N be the Galois closure of L/\mathbb{Q} . Since $L \otimes \mathbb{Q}_p$ has a component equal to \mathbb{Q}_p , we have $\text{Gal}(N_P/\mathbb{Q}_p) \leq \mathcal{S}_3$ for any prime P of N above p .

Let \tilde{K} be the Galois closure of K/\mathbb{Q} contained in N/\mathbb{Q} and let \mathfrak{p} be any prime above p in \tilde{K} . The Galois group of N/\tilde{K} is the unique order four normal subgroup V of \mathcal{S}_4 . Therefore, if P is any prime of N above \mathfrak{p} , we have $\text{Gal}(N_P/\tilde{K}_{\mathfrak{p}}) \leq V$. However, the subgroups V and \mathcal{S}_3 of \mathcal{S}_4 intersect trivially. As a result, we have $\text{Gal}(N_P/\tilde{K}_{\mathfrak{p}}) = 0$ and \mathfrak{p} splits completely in N/\tilde{K} .

We therefore see that $N \otimes \mathbb{Q}_p = (\tilde{K} \otimes \mathbb{Q}_p)^4$. Taking $\text{Gal}(N/L)$ invariants, we get that $L \otimes \mathbb{Q}_p = (K \otimes \mathbb{Q}_p) \oplus \mathbb{Q}_p$. \square

Lemma 4.1 motivates the following definition. Given a rank 3 \mathbb{Q}_p -algebra R_p , we define a rank 4 \mathbb{Q}_p -algebra \widehat{R}_p as $\widehat{R}_p = R_p \oplus \mathbb{Q}_p$. We then get the following corollary.

Proposition 4.2. *Let $\Sigma = (\Sigma_p)_{p \in S}$ where each Σ_p is a set of rank 3 \mathbb{Q}_p -algebras. Set $\tilde{\mu}(\Sigma) = \prod_{p \in S} \mu_p(\widetilde{\Sigma_p}) / \mu_p(\Sigma_p)$, where $\widetilde{\Sigma_p} = \{\widetilde{R_p} : R_p \in \Sigma_p\}$. Then*

- (1) *the average size of $\text{Cl}(K)_S[2]$ as K ranges over totally real \mathcal{S}_3 -cubic fields with $K \otimes \mathbb{Q}_p \in \Sigma_p$ for all $p \in S$ is equal to $1 + \frac{1}{4}\tilde{\mu}(\Sigma)$*
- (2) *the average size of $\text{Cl}(K)_S[2]$ as K ranges over complex \mathcal{S}_3 -cubic fields with $K \otimes \mathbb{Q}_p \in \Sigma_p$ for all $p \in S$ is equal to, and $1 + \frac{1}{2}\tilde{\mu}(\Sigma)$*
- (3) *the average size of $\text{Cl}(K)_S^+[2]$ as K ranges over totally real \mathcal{S}_3 -cubic fields with $K \otimes \mathbb{Q}_p \in \Sigma_p$ for all $p \in S$ is equal to $1 + \tilde{\mu}(\Sigma)$.*

Proof. By combining Corollary 2.8 with Lemma 4.1, we have

$$\sum_{\substack{K \text{ cubic,} \\ K \otimes \mathbb{Q}_p \simeq R_p \text{ for all } p \in S \\ 0 < \text{Disc}(K) < X}} |\text{Cl}(K)_S[2]| = M^{(3)}(X, \Sigma) + N^{(4)}(X, \widetilde{\Sigma}).$$

Part (i) then follows from Theorems 3.1 and 3.2. Part (ii) follows from an essentially identical calculation.

For part (iii), again by Corollary 2.8 combined with Lemma 4.1, we have

$$\sum_{\substack{K \text{ cubic,} \\ K \otimes \mathbb{Q}_p \simeq R_p \text{ for all } p \in S \\ 0 < \text{Disc}(K) < X}} |\text{Cl}(K)_S^+[2]| = M^{(3)}(X, \Sigma) + N^{(4)}(X, \widetilde{\Sigma}) + N^{(0)}(X, \widetilde{\Sigma}).$$

The result then follows from Theorems 3.1 and 3.2. \square

The quotients $\mu_p(\widetilde{\Sigma_p}) / \mu_p(\Sigma_p)$ have a nice formula when either $\Sigma_p = \{R_p\}$ or Σ_p contains all rank 3 \mathbb{Q}_p -algebras (up to isomorphism).

Lemma 4.3. *If $\Sigma_p = \{R_p\}$, then $\mu_p(\widetilde{\Sigma_p}) / \mu_p(\Sigma_p) = 2^{-(r_p-1)}$, where R_p is the number of irreducible components of R_p .*

Proof. Since $\widetilde{R_p} = R_p \oplus \mathbb{Q}_p$, we have $\text{Disc}_p(\widetilde{R_p}) = \text{Disc}_p(R_p)$, so $\mu_p(\{\widetilde{R_p}\}) / \mu_p(\{R_p\}) = |\text{Aut}(R_p)| / |\text{Aut}(\widetilde{R_p})| = n! / (n+1)! = 1 / (n+1)$, where n is the number of components of R_p equal to \mathbb{Q}_p . Examination shows that this is equal to $2^{-(r_p-1)}$. \square

Lemma 4.4. *If Σ_p contains all rank 3 \mathbb{Q}_p -algebras, then $\mu_p(\widetilde{\Sigma_p}) / \mu_p(\Sigma_p) = \frac{1}{2} \left(1 + \frac{p^2+4}{4(p^2+p+1)} \right)$.*

Proof. By Lemma 4.3, we have $\mu_p(\{\widetilde{R_p}\}) / \mu_p(\{R_p\}) = 2^{-(r-1)}$ for each R_p with r irreducible components. Letting $\Sigma_{p,r}$ denote the set of all rank 3 \mathbb{Q}_p -algebras with r irreducible components, we get

$$\mu_p(\widetilde{\Sigma_p}) = \sum_{R_p \in \Sigma_p} \mu_p(\widetilde{\Sigma_{p,r}}) = \sum_{r=1}^3 2^{-(r-1)} \mu_p(\Sigma_{p,r}).$$

Calculating $\mu_p(\Sigma_{p,r})$ for each $r \in \{1, 2, 3\}$, we find

$$\mu_p(\Sigma_{p,1}) = \frac{p^3 - p^2 + 3p - 3}{3p^3}, \mu_p(\Sigma_{p,2}) = \frac{p^2 + p - 2}{2p^2}, \text{ and } \mu_p(\Sigma_{p,3}) = \frac{p-1}{6p}.$$

As a result, we get $\mu_p(\widetilde{\Sigma_p}) = \frac{p^3 - p^2 + 4p - 8}{8p^3}$, so

$$\mu_p(\widetilde{\Sigma_p}) / \mu_p(\Sigma_p) = \frac{p^3 - p^2 + 4p - 8}{8p^3} \cdot \frac{p^3}{p^3 - 1} = \frac{5p^2 + 4p + 8}{8(p^2 + p + 1)} = \frac{1}{2} \left(1 + \frac{p^2 + 4}{4(p^2 + p + 1)} \right).$$

□

We are now able to prove Theorems 2 and 4.

Proof of Theorem 2. For each $p \in S$, let Σ_p be the set of all rank 3 \mathbb{Q}_p algebras. The result then follows from combining Proposition 4.2 with Lemma 4.4. □

Proof of Theorem 4. For each $p \in S$, let $\Sigma_p = \{R_p\}$. By Lemma 4.3, we have $\mu_p(\widetilde{\Sigma_p})/\mu_p(\Sigma_p) = 2^{-(r_p-1)}$ for each $p \in S$. Letting $r = \sum_{p \in S} (r_p - 1)$, we have $\widetilde{\mu}(\Sigma) = 2^{-r}$ and the result follows from Proposition 4.2. □

Corollary 3 will now follow from Theorem 2.

Proof of Corollary 3. Let λ be the proportion of totally real \mathcal{S}_3 -cubic fields K having $\text{Cl}(K)_S^+[2] = 0$. We then have $\text{Avg}(\text{Cl}(K)_S^+[2]) \geq \lambda + 2 \cdot (1 - \lambda) = 2 - \lambda$. By Theorem 2, we have

$$\text{Avg}(\text{Cl}(K)_S^+[2]) = 1 + \frac{1}{2^{|S|}} \prod_{p \in S} \left(1 + \frac{p^2 + 4}{4(p^2 + p + 1)} \right),$$

so it follows that

$$(6) \quad \lambda \geq 1 - \frac{1}{2^{|S|}} \prod_{p \in S} \left(1 + \frac{p^2 + 4}{4(p^2 + p + 1)} \right).$$

If S is non-empty, then the right-hand side of (6) is at least $5/14$, proving (i).

To see (ii), observe that the kernel of the surjection $\text{Cl}(K)_S^+ \rightarrow \text{Cl}(K)_S$ is a subgroup of $\text{Cl}(K)_S^+$ having order at most two. As a result, if $\text{Cl}(K)_S^+$ has odd order, then $\text{Cl}(K)_S^+ = \text{Cl}(K)_S$. The result then follows from (i).

Finally, we note that K has S_K -units of all signatures if and only if $\text{Cl}(K)_S^+ = \text{Cl}(K)_S$. Part (iii) of the corollary therefore follows from (ii). □

5. AVERAGES FOR $\mathcal{O}_{K,S}^\times/(\mathcal{O}_{K,S}^\times)^2$ AND $\text{Sel}_2^S(K)$

We now wish to consider the average size of $\mathcal{O}_{K,S}^\times/(\mathcal{O}_{K,S}^\times)^2$.

Define a function ν_S on \mathcal{S}_3 -cubic fields by $\nu_S(K) = \sum_{p \in S} r_p(K)$, where $r_p(K)$ is the number of irreducible components of $K \otimes \mathbb{Q}_p$. By Dirichlet's unit theorem, if K is a real cubic field, then $\mathcal{O}_{K,S}^\times/(\mathcal{O}_{K,S}^\times)^2$ has rank $\nu_S(K) + 3$ as an \mathbb{F}_2 -vector space. Similarly, if K is a complex cubic field, then $\mathcal{O}_{K,S}^\times/(\mathcal{O}_{K,S}^\times)^2$ has rank $\nu_S(K) + 2$.

We now proceed to prove Theorem 6.

Proof of Theorem 6. We only consider the totally real case, since the proof in the complex case is nearly identical. We proceed by induction on the set S . For the base case, consider the case where S contains a single prime p . By Dirichlet's unit theorem, we have

$$(7) \quad \text{Avg}(\mathcal{O}_{K,S}^\times/(\mathcal{O}_{K,S}^\times)^2) = 8 \cdot \sum_{n=1}^3 2^n \cdot \text{Prob}(K \otimes \mathbb{Q}_p \text{ has } n \text{ irreducible components}).$$

By Theorem 3.2, we have

$$(8) \quad \text{Prob}(K \otimes \mathbb{Q}_p \text{ has } r \text{ irreducible components}) = \frac{\mu_p(\Sigma_{p,r})}{\mu_p},$$

where $\Sigma_{p,r}$ is as in the proof of Lemma 4.4 and $\mu_p = 1 - \frac{1}{p^3}$. Evaluating (8), we get

$$\begin{aligned}
 (9) \quad & \sum_{n=1}^3 2^n \cdot \text{Prob}(K \otimes \mathbb{Q}_p \text{ has } n \text{ irreducible components}) \\
 &= \frac{p^3}{p^3 - 1} \left(2 \cdot \frac{p^3 - p^2 + 3p - 3}{3p^3} + 4 \cdot \frac{p^2 + p - 2}{2p^2} + 8 \cdot \frac{p - 1}{6p} \right) \\
 &= 4 - \frac{2}{p^2 + p + 1} = 2 \cdot \left(2 - \frac{1}{p^2 + p + 1} \right).
 \end{aligned}$$

Combining (7) with (9), we get

$$\text{Avg}(\mathcal{O}_{K,S}^\times / (\mathcal{O}_{K,S}^\times)^2) = 2^{|S|+3} \left(2 - \frac{1}{p^2 + p + 1} \right).$$

For the inductive step, let $S' = S \cup \{p\}$ for some prime $p \notin S$. Applying Dirichlet's unit theorem once again, we have

$$\begin{aligned}
 (10) \quad & \text{Avg}(\mathcal{O}_{K,S'}^\times / (\mathcal{O}_{K,S'}^\times)^2) = \\
 & \sum_s \left(2^{s+3} \cdot \text{Prob}(\nu_S(K) = s) \cdot \sum_{n=1}^3 2^n \cdot \text{Prob}(K \otimes \mathbb{Q}_p \text{ has } n \text{ irred. components} \mid \nu_S(K) = s) \right)
 \end{aligned}$$

By Theorem 3.2, the behavior of $K \otimes \mathbb{Q}_p$ is independent of $K \otimes \mathbb{Q}_{p'}$ for all $p' \in S$, and therefore independent of $\nu_S(K)$. As a result, we may switch the order of summation and get

$$\begin{aligned}
 (11) \quad & \text{Avg}(\mathcal{O}_{K,S'}^\times / (\mathcal{O}_{K,S'}^\times)^2) = \\
 &= \left(\sum_{n=1}^3 2^n \cdot \text{Prob}(K \otimes \mathbb{Q}_p \text{ has } n \text{ irred. components}) \right) \sum_s 2^{s+3} \cdot \text{Prob}(\nu_S(K) = s) \\
 &= \left(\sum_{n=1}^3 2^n \cdot \text{Prob}(K \otimes \mathbb{Q}_p \text{ has } n \text{ irred. components}) \right) \text{Avg}(\mathcal{O}_{K,S}^\times / (\mathcal{O}_{K,S}^\times)^2)
 \end{aligned}$$

By the inductive hypothesis, we have

$$(12) \quad \text{Avg}(\mathcal{O}_{K,S}^\times / (\mathcal{O}_{K,S}^\times)^2) = 2^{|S|+3} \prod_{p \in S} \left(2 - \frac{1}{p^2 + p + 1} \right)$$

and by (9), we have

$$(13) \quad \sum_{n=1}^3 2^n \cdot \text{Prob}(K \otimes \mathbb{Q}_p \text{ has } n \text{ irreducible components}) = 2 \cdot \left(2 - \frac{1}{p^2 + p + 1} \right).$$

The result follows from combining (11), (12), and (13). \square

6. AVERAGES OF SELMER GROUPS

In the setting of Theorem 4, we are given a fixed rank 3 \mathbb{Q}_p -algebra R_p for each $p \in S$ and we are able to compute the average size of $\text{Cl}(K)_S[2]$ as we range over \mathcal{S}_3 -cubic fields K with a fixed signature such that $K \otimes \mathbb{Q}_p \simeq R_p$ for every $p \in S$.

As seen in Theorem 4, the average size does not depend on the collection of R_p , only on $r = \sum_{p \in S} (r_p - 1)$. As a result, we get the following:

Proposition 6.1. *Let s be an integer with $|S| \leq s \leq 3|S|$.*

- (i) *The average size of $\text{Cl}(K)_S[2]$ as K ranges over totally real \mathcal{S}_3 -cubic fields with $\nu_S(K) = s$ is equal to $1 + 2^{-(s-|S|+3)}$ and*
- (ii) *the average size of $\text{Cl}(K)_S[2]$ as K ranges over complex \mathcal{S}_3 -cubic fields with $\nu_S(K) = s$ is equal to $1 + 2^{-(s-|S|+2)}$.*

Proof. This follows directly from Theorem 4 combined with the fact that $\sum_{p \in S} (r_p - 1) = \nu_S(K) - |S|$. \square

We then get the following corollary:

Corollary 6.2. *Let s be an integer with $|S| \leq s \leq 3|S|$. When ordered by absolute discriminant,*

- (i) *the average size of $\text{Sel}_2^S[2]$ as K ranges over totally real cubic fields with $\nu_S(K) = s$ is equal to $2^{(s+3)} + 2^{|S|}$ and*
- (ii) *the average size of $\text{Sel}_2^S[2]$ as K ranges over complex cubic fields with $\nu_S(K) = r$ is equal to $2^{(s+2)} + 2^{|S|}$.*

Proof. For each K , we have $|\text{Sel}_2^S(K)[2]| = |\mathcal{O}_{K,S}^\times / (\mathcal{O}_{K,S}^\times)^2| \cdot |\text{Cl}(K)_S[2]|$. Since $|\mathcal{O}_{K,S}^\times / (\mathcal{O}_{K,S}^\times)^2|$ is constant over all totally real (resp. complex) K with $\nu_S(K) = s$, we therefore have

$$\text{Avg}(|\text{Sel}_2^S(K)[2]|) = \text{Avg}(|\mathcal{O}_{K,S}^\times / (\mathcal{O}_{K,S}^\times)^2| \cdot |\text{Cl}(K)_S[2]|) = |\mathcal{O}_{K,S}^\times / (\mathcal{O}_{K,S}^\times)^2| \text{Avg}(|\text{Cl}(K)_S[2]|).$$

The results then follow from Proposition 6.1. \square

We are now able to prove Theorem 5.

Proof of Theorem 5. Once again, we only consider the totally real case, since the proof in the complex case is nearly identical. For each integer s with $|S| \leq s \leq 3|S|$, define $\rho(s)$ to be the proportion of \mathcal{S}_3 -cubics K such that $\nu_S(K) = s$. By Corollary 6.2, we then have

$$\begin{aligned} (14) \quad \text{Avg}(|\text{Sel}_2^S(K)|) &= \sum_{s=|S|}^{3|S|} (2^{(s+3)} + 2^{|S|}) \cdot \rho(s) = \sum_{s=|S|}^{3|S|} 2^{|S|} \cdot \rho(s) + \sum_{s=|S|}^{3|S|} 2^{s+3} \cdot \rho(s) \\ &= 2^{|S|} + \sum_{s=|S|}^{3|S|} 2^{s+3} \cdot \rho(s). \end{aligned}$$

We then observe that by Dirichlet's unit theorem,

$$(15) \quad \sum_{s=|S|}^{3|S|} 2^{s+3} \cdot \rho(s) = \text{Avg}(|\mathcal{O}_{K,S}^\times / (\mathcal{O}_{K,S}^\times)^2|),$$

and by Theorem 6,

$$(16) \quad \text{Avg}(|\mathcal{O}_{K,S}^\times / (\mathcal{O}_{K,S}^\times)^2|) = 2^{|S|+3} \prod_{p \in S} \left(2 - \frac{1}{p^2 + p + 1} \right).$$

Combining (14), (15), and (16), we therefore get that

$$\text{Avg}(|\text{Sel}_2^S(K)|) = 2^{|S|} + 2^{|S|+3} \prod_{p \in S} \left(2 - \frac{1}{p^2 + p + 1} \right).$$

\square

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